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ABSTRACT

This paper addresses the problem of how to place students in a sequence of hierarchically related courses from an (empirical) Bayesian point of view. Based on a minimal set of assumptions, it is shown that optimal mastery rules for the courses are always monotone and a nonincreasing function of the scores on the placement test. On the other hand, placement rules are not generally monotone but have a form depending on the specific shape of the probability distributions and utility functions in force. The results are further explored for a class of linear utility functions. Numerous illustrations and tables present data and statistical analysis. (Contains 20 references.) (Author/TS)

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Research Report 95-01

Wim J. van der Linden

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Some Decision Theory for Course Placement

Wim J. van der Linden

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Abstract

The problem of how to place students in a sequence of hierarchically related courses is addressed from an (empirical) Bayesian point of view. Based on a minimal set of assumptions, it is shown that optimal mastery rules for the courses are always monotone and a nonincreasing function of the scores on the placement test. On the other hand, placement rules are not generally monotone but have a form depending on the specific shape of the probability distributions and utility functions in force. The results are further explored for a class of linear utility functions.

Some Decision Theory for Course Placement

A course placement problem is met when students with different aptitudes enter an educational program offering classes in the same domain at several entrance levels. The typical solution of the problem is to administer an aptitude test to the students to decide at which level they should start. It is the purpose of this paper to demonstrate how (Bayesian) decision theory could be used to optimize such placement decisions.

For simplicity, the case of two courses is considered where the first course has a lower entrance level than the second course. It is assumed that an aptitude test is administered to decide whether the students have to take the first course or can go straight to the second course. In addition, it is assumed that the first course ends with a mastery test to decide whether those students taking the first course have learned its content and can be admitted to the second course. A flowchart of the problem displaying the temporal relationship between the tests and courses if given in Figure 1.

[Figure 1 about here]

To derive optimal decision rules for the placement and mastery decisions, (Bayesian) decision theory is a natural framework. Examples of a decision-theoretic treatment of the mastery problem are found in Hambleton and Novick (1973), Huynh (1976, 1977, 1982), Huynh and Perney (1979), van der Linden (1980, 1990), and van der Linden and Mellenbergh (1977). The placement problem has been addressed in Sawyer (submitted) and van der Linden (1981). The novelty in this paper is that optimal rules for the two decisions problems are

derived simultaneously. The advantages of a simultaneous approach have been spelled out in Vos (1994,1995). For the present problem, they will be discussed later in this paper.

Notation

Sampling of students from a population P is assumed. The observed scores on the aptitude or placement test, administered before the first course, will be denoted by a random variable X. Likewise, the observed score on the mastery test is denoted by Y whereas the true score on this test is denoted by T. For ease of exposition, the variables are assumed to be continuous.

For each possible value of the observed score, the decision to assign the students either to Course 1 or to Course 2 has to be made. A placement rule can therefore be denoted either by the set of x values A_1={x:Course 1} or its complement A_2={x:Course 2}. Likewise, a mastery rule can be represented by the set $B_1(x)$={y:x,Fail} or its complement $B_2(x)$={y:x,Pass}. Note that the mastery rule is allowed to depend on the score on the placement test, X=x. This type of mastery rule presents the most general approach. For example, it is possible that for candidates in Course 1 with a high score on the placement test a different mastery rule is optimal than for candidates with low scores. The idea to allow mastery rules to depend on the scores on a previous placement test is congenial with the idea developed in Bayesian statistics that the use of collateral information generally improves the quality of decision making. On the other hand, as will be shown below, constraining optimal rules $B_1(x)$ or $B_2(x)$ to be the same sets of y values for each possible value x requires additional conditions on the test score distributions.

Distributional Assumptions

It is assumed that if all students in P are assigned to Course 1, the trivariate distribution of their scores (X,Y,T) can be represented by a density function $f_1(x,y,t)$. If, on the other hand, all students are assigned directly to Course 2, the distribution is assumed to be represented by a density function $f_2(x,y,t)$. For future reference, it is understood that the conditional distributions of T given (X=x,Y=y), T given X=x, Y given X=x, and (Y,T) given X=x are denoted by $p_i(t|x,y)$, $q_i(t|x)$, $h_i(y|x)$, and $r_i(y,t \mid x)$, i=1,2, respectively. The marginal distribution of X is denoted by $g_i(x)$. Without loss of generality, it is assumed that all density functions are nonnegative everywhere. As usual, cumulative distribution functions will be denoted by capitals.

The following assumptions about $f_i(x,y,t)$ seem plausible:

$g_i(x)=g(x)$, i=1,2; **(1)**

$P_i(t|x,y)$ is decreasing in x and y for all values of t, i=1,2; **(2)**

$H_i(y|x)$ is decreasing in x for all values of y, i=1,2; **(3)**

$Q_i(t|x)$ is decreasing in x for all values of t, i=1,2. **(4)**

The first assumption is realized if the placement test is administered before either course is taught--a condition automatically met in regular course placement problems. The last three assumptions require the conditional distributions of the observed and true mastery test scores to be stochastic increasing in their

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conditioning variables. These conditions can be expected to be met if the placement and mastery tests are well designed, that is, if they are constructed such that high scores on the placement test tend to coincide with high observed and true scores on the mastery test. In van der Linden and Vos (in press), it is proved that conditions as in (4) follow from the assumptions in (2)-(3).

Several other assumptions could be specified to define a course placement problem with well-designed tests and courses. For example, for the conditional cumulative distributions functions of (Y,T) given X=x, $R_i(y,t|x)$, it seems reasonable to expect that:

$R_i(y,t|x)$ increases in i for all values of x, y, and t. **(5)**

This condition states that the probability of high observed and true scores on the mastery test given a score on the placement test is larger for students assigned to Course 1 than for those assigned directly to Course 2. In other words, no matter the aptitude scores of the students, following Course 1 will tend to have a positive effect on their observed and true mastery of its subject matter. However, this and other obvious properties are not needed to derive the results presented in this paper. For a more general treatment of the properties of stochastic order in multivariate distributions of test scores, see van der Linden (submitted).

Utility Structure

The usual approach is to define the utilities involved in mastery decisions as a function of the true score underlying the mastery test. Since mastery decisions are

made only for students assigned to Course 1, it seems logical to also view the utilities involved in the outcomes of this decision as a function of the true mastery score. As for Course 2, the position is taken that the decision to assign students directly to this course can be viewed as an advanced mastery decision based on the aptitude test as a predictor of the students' mastery scores after Course 1. It follows that the true mastery score can be used as the common criterion to measure the utilities involved in all decisions in the simultaneous placement-mastery problem addressed here. More in particular, the following utility functions are defined:

$$\begin{aligned} &\text{Assignment to Course 1 } (x \in A_1) \text{ and Fail } (y \in B_1(x)): && u_1(t); \\ &\text{Assignment to Course 1 } (x \in A_1) \text{ and Pass } (y \in B_2(x)): && u_2(t); \\ &\text{Assignment to Course 2 } (x \in A_2): && u^*(t). \end{aligned} \tag{6}$$

As for the shape of the utility functions, it is assumed that:

$u_1(t)$ is monotonically decreasing in t; (7)

$u_2(t)$ and $u^*(t)$ are monotonically increasing in t. (8)

Condition (7) can be defended pointing at the fact that the decision to have a student fail after Course 1 has less utility, the higher the true mastery level of the student. The reverse is true for the other two possible decisions; both for the actual mastery and the "advanced mastery" decision, it should hold that the

outcome has larger utility, the higher the true mastery level of the student. However, the actual mastery decision involves the additional costs of teaching Course 1 which are missed for the "advanced mastery" decision. For traditional group-based instruction, these costs can be assumed to be a constant independent of t. Hence,

$$u^*(t) \geq u_2(t) \text{ for all values of } t. \qquad (9)$$

As yet, no parametric form will be assumed for the utility functions. The intention is first to further explore the features of the problem as well as the shape of its optimal decision rules. Then a parametric form for the utility functions will be introduced. In doing so, two additional assumptions about the nature of the difference between $u_2(t)$ and $u^*(t)$ will be made.

Formalization of the Problem

The condition on the utility functions in (9) suggests that if utilities were the only concern, the best decision would be to always assign the students to Course 2. This action represents the largest utility since the constant costs involved in teaching the first course are missed. However, if the decisions had to be based only on the test score distributions, the best decision would always be to assign students to Course 1. As the condition in (5) shows, no matter the score of the students on the placement test, the probability of having high observed or true mastery of the subject matter taught in this course increases when attending it. Therefore, if mastery of Course 1 is a prerequisite for success in Course 2,

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students who attend Course 1 will tend to be more successful in Course 2.

Obviously, a criterion is needed to solve this dilemma between utility and probability. In this paper, the Bayes criterion of maximum expected utility is used. For the utility functions and distributions defined above, the expected utility associated with the simultaneous problem considered in this paper can be written as:

$$E[U(A_1,B_1(x))] = \int_{A_1}\{\int_{B_1(x)}\int_{R} u_1(t)f_1(x,y,t)\,dt\,dy + \int_{B_2(x)}\int_{R} u_2(t)f_1(x,y,t)\,dt\,dy\}\,dx + \int_{A_2}\int_{R}\int_{R} u^*(t)f_2(x,y,t)\,dt\,dy\,dx. \quad (10)$$

It is assumed that the integrals in (10) converge. Formally, the problem is now to find sets A_1 and $B_1(x)$ for which the expression in (10) is maximal.

Moving to posterior densities and taking expectations, (10) can be written as:

$$E[U(A_1,B_2(x))] = \int_{A_1}\{E_1[u_1(T)|x] - E_2[u^*(T)|x] + \int_{B_2(x)} E_1[u_2(T) - u_1(T)|x,y]h_1(y|x)\,dy\}g(x)\,dx + E_2[u^*(T)]. \quad (11)$$

where the subscripts at the expectation signs index the distributions with respect to which the expectation is taken. The problem is now to find sets A_1 and $B_2(x)$ for which (11) is maximal. Note that the last term of (11) is a constant independent of

A_1 and $B_2(x)$ and may be ignored. Since the sample spaces of X and Y are not constrained in any way, the optimal sets can be found by first maximizing the second term over all possible sets of y values and next maximizing the sum of the first two terms over all possible sets of x values.

General Solution to the Problem

The general results in this paper are presented in the form of two propositions. The first proposition states that under the conditions given above the optimal mastery rule is always a cutting score on the mastery test score Y which is a decreasing function of the placement test score X=x. The second proposition states that the form of the optimal placement rule does not necessarily have the form of a cutting score on the placement test.

Proposition 1. Under conditions (1)-(3) on the posterior distributions of T given (X=x,Y=y) and Y given X=x, it holds for the utility functions in (7)-(8) that the optimal sets $B_2(x)$ are intervals $[y_c(x), \infty)$, where $y_c(x)$ is a decreasing function of x.

Proof. From (7)-(8) it follows that $u_2(t)-u_1(t)$ is increasing in t whereas the condition in (2) states that the distribution of T given (X=x,Y=y) is stochastic increasing in the two conditioning variables. From a well-known theorem for such distributions (Lehmann, 1986, p 116), it follows that $E_1[u_2(T)-u_1(T) \mid x,y]$ increases in x and y. Let $\tau(x,y)$ denote this expectation, and consider the relation $\{(x,y) \mid \tau(x,y)=0\}$. This relation defines a function $y=y_c(x)$ which is decreasing in y (van der Linden & Vos,

in press, Lemma 4). Hence, for each x there exists a value $y_c(x)$ such that $\tau(x,y)$ is nonnegative for $y \geq y_c(x)$ but negative for $y < y_c(x)$. Since the integral of $\tau(x,y)$ over $h_1(y|x)$ is maximal if it is taken over all values of x for which the integrand is nonnegative, it follows that the optimal sets $B_2(x)$ are the intervals $(y_c(x), \infty]$. ■

Proposition 2. Under the conditions stated above it does not hold generally that the optimal set A_1 is monotone, that is, there always exists a cutting score x_c such that $A_1 = (-\infty, x_c]$ is optimal.

Proof. From (4) and (7) it follows that $E_1[u_1(T)|x]$ decreases in x whereas (4) and (8) imply that $E_2[u^*(T)|x]$ increases in x. Therefore, the integrand in the first term in (11) decreases in x. Following an argument in van der Linden and Vos (in press), it is now proved that the integrand in the second term in (11) is nondecreasing in x. Substituting $B_2(x) = [y_c(x), \infty)$ into (11), since $\tau(x,y)$ is nonnegative for $y \geq y_c(x)$ and $y_c(x)$ is decreasing in x, it holds for any $x_2 > x_1$ that

$$\int_{y_c(x_2)}^{\infty} \tau(x_2,y) h_1(y|x_2) dy - \int_{y_c(x_1)}^{\infty} \tau(x_1,y) h_1(y|x_1) dy >$$

$$\int_{y_c(x_1)}^{\infty} \tau(x_2,y) h_1(y|x_2) dy - \int_{y_c(x_1)}^{\infty} \tau(x_1,y) h_1(y|x_1) dy >$$

$$\int_{y_c(x_1)}^{\infty} \tau(x_1,y) [h_1(y|x_2) - h_1(y|x_1)] dy =$$

$$\int_{-\infty}^{\infty} \varphi(y) [h_1(y|x_2) - h_1(y|x_1)] dy. \qquad (12)$$

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where $\varphi(y) \equiv I_{[y_C(x_1),\infty)}(y)\tau(x_1,y)$, and $I_{[y_C(x_1),\infty)}(y)$ is an indicator function which takes the value 1 if $y \in [y_C(x_1),\infty)$ and the value 0 otherwise. By definition, $\varphi(y)$ is a nondecreasing function of y, and from (3) it follows that the second term in (11) is nondecreasing indeed. Now it depends on the rate of change of the first and second terms in (11), and hence on the specific form of the utility functions and score distributions whether the integrand in the integral over x in (11) is a monotonically decreasing function of x. Therefore, the optimal set A_1 does not generally take a monotone form. ■

Proposition 2 implies that for some utility functions and distributions, an optimal set A_1 may consists of two or more non-adjacent intervals of placement test scores. Generally, it does not hold that if a student with a certain aptitude is assigned to Course 1, the same should apply for all students with lower aptitudes. The result goes against the prevailing practice of using placement rules in the form of a single cutting score on a test. The explanation of this "anomaly" is the dilemma between utility and probability referred to in the previous section. Utilities and probabilities are different quantities. If for a given aptitude level it is known that the expected increase in mastery due to attending Course 1 offsets the increase in utility involved in directly attending Course 2, no inferences as to other aptitude levels are possible without knowing the specific forms of the utility functions and probability distributions.

Calculation of Optimal Rules

From the proof of Proposition 1 it follows that the optimal mastery function is the decreasing function $y=y_C(x)$ defined by

$$E_1[u_2(T) - u_1(T) | x,y] = 0. \tag{13}$$

Substituting $y_c(x)$ into (11), maximal expected utility is obtained if A_1 is taken to be the set of x values for which the integrand of the outer integral is nonnegative. That is, the optimal set A_1 is given by:

$$A_1 = \{x: E_1[u_1(T)|x] - E_2[u^*(T)|x] + \int_{y_c(x)} E_1[u_2(T) - u_1(T)|x,y] h_1(y|x)\,dy \geq 0\}. \tag{14}$$

The optimal set in (14) can be found using a suitable method of numerical integration. As in practice observed test scores are always discrete, a simple approach is to substitute all possible test scores into the expression in (14) to find the subset for which it is nonnegative. However, to be able to do so, the regression functions in (14) must be estimated. The following section shows how to deal with the problem if linear utilities and regression functions can be assumed to hold.

Linear Utility Structure

For the utility functions in (6), the following linear structure is adopted:

$$u_1(t) = b_1(t_c - t) + a_1, \quad b_1 > 0. \tag{15}$$

$$u_2(t) = b_2(t - t_c) + a_2, \quad b_2 > 0. \tag{16}$$

$$u^*(t) = b^*(t-t_c) + a^*, \quad b^* > 0, \tag{17}$$

where t_c is assumed to be a threshold value on the true mastery scale which separates true masters from true nonmasters. If such a threshold does not exist, the above functions should be reparameterized absorbing t_c into the intercept parameter. Note that the slope parameters are required to be larger than zero to meet the conditions in (7)-(8). Also, generally, a_1, a_2, $a^* < 0$ because these utility constants represent the constant parts of the costs involved in teaching Course 1 and/or Course 2. It is reminded that this paper views the decision to assign students to Course 2 as an advanced mastery decision. Hence, utility functions (16) and (17) have the same parameter structure but may have different values for their parameters to deal with the specifics of the application. An example of the use of these utility functions will be given later in this paper. Techniques to elicit utility functions for placement decisions have been addressed in Sawyer (1994, April).

Substituting utility functions (15)-(17) into the expression for the expected utility in (11) and using the result in Proposition 1:

$$E[U(A_1, y_C(x))] = \int_{A_1} \{(b_1 + b^*)t_C + (a_1 - a^*) - b_1 E_1(T|x) - b^* E_2(T|x) + \int_{y_C(x)}^{\infty} [-(b_1 + b_2)t_C + (a_2 - a_1) + (b_1 + b_2)E_1(T|x,y)]h_1(y|x)\,dy\}g(x)\,dx. \tag{18}$$

From the general solution in (13) it follows that the optimal mastery function is defined by:

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$$-(b_1+b_2)t_C + (a_2-a_1) + (b_1+b_2)E_1(T|x,y) = 0, \quad (19)$$

whereas (14) implies that the following set of x values is optimal:

$$A_1 = \{x: (b_1+b^*)t_C + (a_1-a^*) - b_1E_1(T|x) - b^*E_2(T|x)$$

$$+ \int_{y_C(x)}^{\infty} [-(b_1+b_2)t_C + (a_2-a_1) + (b_1+b_2)E_1(T|x,y)h_1(y|x)dy \geq 0\}. \quad (20)$$

Linear Regression

If linear regression of T on X and Y can be assumed, that is, if it holds that

$$E_1(T|x,y) = \alpha_{TXY1} + \beta_{TXY1}x + \gamma_{TXY1}y, \quad (21)$$

the optimal mastery function is:

$$y_C(x) = [t_C - (a_2-a_1)/(b_1+b_2) - \alpha_{TXY1}]\gamma_{TXY1}^{-1} - (\beta_{TXY1}\gamma_{TXY1}^{-1})x. \quad (22)$$

Note that this function is linear in x with a slope parameter equal to $-\beta_{TXY}\gamma_{TXY}^{-1}$. Since families of conditional distributions with the property in (2) have nonnegative covariances between the random variable and the conditioning variables (van der Linden, submitted), it follows that β_{TXY} and γ_{TXY} are nonnegative, and thus that the function in (22) is nondecreasing indeed.

As classical test theory shows:

$$E_i(T|x) = E_i(Y|x).$$

The former can thus be written as:

$$E_i(T|x) = \alpha_{YX_i} + \beta_{YX_i}x. \tag{23}$$

Substituting (21) and (23) into (20), the optimal set A_1 is:

$$A_1 = \{x: K_1 - (b_1\beta_{YX1} + b^*\beta_{YX2})x + \int_{y_C(x)} [K_2 + (b_1 + b_2)(\beta_{TYX_1}x + \gamma_{TYX_1}y)h_1(y|x)]dy \geq 0\}, \tag{24}$$

with

$$K_1 = b_1(t_C - \alpha_{YX1}) + b^*(t_C - \alpha_{YX2}) + (a_1 - a^*), \tag{25}$$

$$K_2 = (b_1 + b_2)(\alpha_{TXY1} - t_C) + (a_2 - a_1), \tag{26}$$

and where $y_C(x)$ is given by (22).

Classical test theory shows that the regression parameters in (22) and (24)-(26) are simple functions of the mean of Y, the variances of X and Y, the correlation between X and Y, and the reliability of Y. If X and Y are standardized, the relevant expressions are: $\alpha_{YX_i} = 0$, $\beta_{YX_i} = \rho_{XY_i}$, $\alpha_{TXY_i} = 0$, $\beta_{TXY_i} = \rho_{XY_i}(1-\rho_{YY_i})/(1-\rho^2_{XY_i})$, and $\gamma_{TXY_i} = (\rho_{YY_i}-\rho^2_{XY_i})/(1-\rho^2_{XY_i})$ (Lord & Novick, 1968, chaps. 2 and 12).

Substituting the above results into (22), the following form for the optimal

mastery function is obtained:

$$y_C(x) = (t_C - (a_2-a_1)/(b_1+b_2))\left(\frac{1-\rho_{XY1}^2}{\rho_{YY1}-\rho_{XY1}^2}\right) - \left(\frac{\rho_{XY1}(1-\rho_{YY1})}{\rho_{YY1}-\rho_{XY1}^2}\right)x. \quad (27)$$

The same substitution into (24) gives:

$$A_1 = \{x;\ K_1 - K_3 x + \int_{y_C(x)}^{\infty} (K_2 + K_4 x + K_5 y)h_1(y|x)dy \geq 0\}, \quad (28)$$

with

$$K_1 = (b_1 + b^*)t_C + a_1 - a^*, \quad (29)$$

$$K_2 = -(b_1 + b_2)t_C + a_2 - a_1, \quad (30)$$

$$K_3 = b_1\rho_{XY1} + b^*\rho_{XY2}, \quad (31)$$

$$K_4 = (b_1 + b_2)\rho_{XY1}(1 - \rho_{YY1})/(1 - \rho_{XY1}^2), \quad (32)$$

$$K_5 = (b_1 + b_2)(\rho_{YY1} - \rho_{XY1}^2)/(1 - \rho_{XY1}^2). \quad (33)$$

To use the above decision rules the conditional density functions $h_1(y \mid x)$ have to be estimated from the data. Fitting a well-chosen parametric form for these densities is an obvious approach to the estimation problem. If $h_1(y \mid x)$ is chosen to be the normal density function, an appropriate change of variable in the integral and use of the fact that for the c.d.f., $\Phi(u)$, it holds that $|\Phi(u)-[1+\exp(1.7u)]^{-1}|$

< 0.01 uniformly in u (Lord & Novick, 1968, sect. 17.2) show that (28) is equal to:

$$A_1 = \{x: K_1 - K_3 x + (K_2 + K_6 x)[1 + \exp(1.7\frac{y_C(x)-\rho_{XY1}x}{1-\rho^2_{XY1}})]^{-1} + K_7 \exp(-\frac{1}{2}(\frac{y_C(x)-\rho_{XY1}x}{1-\rho^2_{XY1}})^2) \geq 0\}, \quad (34)$$

with

$$K_6 = K_4 + \rho_{XY1} K_5, \quad (35)$$

$$K_7 = (2\pi)^{-1/2}(1-\rho^2_{XY1})K_5. \quad (36)$$

Obviously, the second and the fourth term in left-hand side of the inequality in (34) are monotonically decreasing in x. Since the slope of $y_C(x)$ in (27) can never be positive, the second factor of the third term is also monotonically decreasing in x. However, the first factor of this term is increasing in x. Thus it depends on the relative acceleration of these two factors whether or not the left-hand side of the inequality is monotonically decreasing in x and allows for an optimal placement rule of the form $A_1 = (-\infty, x_C]$.

Numerical Examples

A few numerical examples were calculated to shed some light on the behavior of the optimal placement rule and mastery function in (34) and (27) for several values of their parameters. The following six cases were addressed:

Case 1: Standard parameter values. Correlations between cognitive predictor variables and college grades typically range from 0.50-0.60 (Etaugh, Etaugh & Hurd, 1972; see also Schoenfeldt & Brush, 1975). A choice of 0.55 for ρ_{XY_1} seems therefore realistic. The same authors found an average reliability for the grade-point average over a large variety of programs equal to 0.689. Since the average number of courses per program was 5.23, the reliability of a single grade is generally lower. On the other hand, tests for remedial courses typically are more carefully constructed than tests for regular courses. Based on these considerations, a choice of $\rho_{YY_1}=0.60$ was made. The cutting score t_c was set equal to 0.5, which is a value $0.5/\sqrt{0.60}$ standard deviations larger than the average score on the mastery test. The values of the utility parameters a_1, a_2, and a^* in (15)-(17) had to be selected relatively to the scale of t. It was assumed that the costs involved in additionally teaching Course 1 were equal for the students who failed and those who passed the mastery test, that is, $a_1=a_2$. On the other hand, sending students directly to Course 2 does involve only the costs of teaching Course 2. Therefore, utility parameter a^* was set larger than the common value of a_1 and a_2. As for the values of the slope parameters b_1 and b_2 in (15)-(16), failing the mastery test for high values of t was assumed to be less serious than passing the test for low values. This choice is justified if the program is structured such that a retest is offered to those students who fail the test but those who pass are allowed to continue without a further check of their true mastery levels. Hence, $b_1<b_2$. As the placement decision is considered as an advanced mastery decision in this paper, the value of its slope parameter was set equal to value for the actual mastery decision, that is, $b_2=b^*$. In summary, the following set of standard values for the parameters in (27) and (34) was assumed:

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$\rho_{XY1} = \rho_{XY2} = 0.55$;

$\rho_{YY1} = 0.60$;

$t_c = 0.50$;

$a_1 = a_2 = -1.0$;

$a^* = -0.50$;

$b_1 = 1.0$;

$b_2 = b^* = 1.2$.

Case 2: Higher costs of teaching Course 1. It was assumed that the constant costs involved in the teaching of Course 1 were higher than the standard value in the previous case. Therefore, the common value of the parameters a_1 and a_2 was lowered to -1.50. All other parameters were kept at their standard values.

Case 3: Utility less sensitive to true mastery level. To simulate the case of the utilities associated with the various decision outcomes being less sensitive to differences between the true mastery scores, the values of the three slope parameters were halved: $b_1 = 0.5$ and $b_2 = b^* = 0.6$. All other parameters were kept at their standard values.

Case 4: Adaptive instruction. If Course 1 is an individualized course in which the intensity of instruction is adapted to the aptitude level of the students, the savings in instructional costs involved in assigning students directly to Courses 2 may decrease as a function of t. To simulate this case, the values of b_1 and b_2 were raised to 1.2 and 1.4, respectively, but all other parameters were kept at their standard values still.

Case 5: Higher predictive validity of placement test. Next, it was supposed that a better placement test was available than the test studied in Case 1. Therefore, ρ_{XY1} was raised to 0.70. The height of this value must be judged relatively to the reliability of the mastery scores which was kept at its standard

value (as were the values of all other parameters).

Case 6: Higher reliability of mastery test. To assess the effect of a higher reliability of the mastery scores on the optimal placement and mastery rules, ρ_{YY1} was raised to 0.55, leaving all other parameters at their standard values.

Figure 2 gives the plot of the optimal mastery function $y_c(x)$ as a function of

[Figure 2 about here]

x. It appears that the function decreases by an amount of 0.74 for an increase of the placement score X equal to one standard deviation. In the second part of Figure 2, $\lambda(x)$ is the left hand-side of the inequality in (34) plotted as a function of x. To find the roots of

[Table 1 about here]

$\lambda(x)=0$, Newton's method, as implemented in Mathematica's FindRoot command (Wolfram, 1993) was used. Table 1 lists the results for the present and the following cases. Only one root was found, at x=0.592. Thus, a monotone placement rule exists for this case.

The plots for all other cases are given in Figure 3. For higher costs of teaching Course 1 (Case 2), the root of $\lambda(x)$ moved down to x=0.094, which, as expected, implies

[Figure 3 about here]

that fewer students are assigned to Course 1. For utilities less sensitive to changes in the true mastery level t (Case 3), $\lambda(x)$ was again found to cross the horizontal axis only once, this time at x=0.092. Making the instruction in Course 1 adaptive to the mastery levels of the students (Case 4) resulted in the existence of two cutting score on the placement, one at x=0.654 and another at x=5.455. Given the distribution of X assumed in this example, the second cutting score can be ignored for practical reasons. A higher validity of the placement test for the examinees assigned to Course 1 (Case 5) resulted in an optimal mastery function running much steeper because the prediction of the mastery based on the scores could now be based on more valid placement scores. The placement rule was nonmonotone again; two cutting scores were found, one at x=0.442 and the other at x=2.778 implying that examinees with lower and very high placement scores had to be assigned to Course 1 and the others to Course 2. Finally, for a higher reliability of the mastery test for the students assigned to Course 1 (Case 6), the relative importance of the placement test as a predictor of the true mastery level went down substantially, and the optimal mastery function became a more horizontal function of the placement score. $\lambda(x)$ now crossed the horizontal axis only at x=0.746.

Concluding Remarks

One of the advantages of optimizing placement and mastery decisions simultaneously is that more realistic utility structures are possible. For example, the point of view taken in this paper that placement decisions are in fact advanced mastery decisions could never have been translated into a utility structure if the

two decisions were treated separately. Another advantage is that the placement scores can be used as collateral information to improve the mastery decisions. The example showed that for a linear utility structure the behavior of the optimal placement rule and mastery function was realistic for various changes in the sensitivity of utility to true mastery level, the adaptivity of the instruction as well as the validity and reliability of the placement and mastery tests, respectively.

The question can be raised under what conditions the optimal cutting score on the mastery test is a constant independent of the score on the placement test rather than the decreasing function defined by (13). From (11) it immediately follows that if

$$h_1(y|x) = h_1(y), \quad \text{for all } x, \tag{37}$$

the inner integral is only a function of y and the mastery rule becomes independent of X=x. If (37) is satisfied, it holds that $\beta_{TXY}=0$, and the optimal mastery function under linear regression in (27) reduces to its first term which is a constant indeed. However, (37) represents the case in which the aptitude test has no predictive validity whatsoever as to the scores on the mastery test. It is doubted whether such tests could ever play a significant role in placement decisions. In a sense, it has thus been demonstrated in this paper that placement decisions can only be based on valid aptitude tests if they involve optimal mastery rules which are a decreasing function of the aptitude scores.

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Table 1

Roots of $\lambda(x)$ for Cases 1 through 6

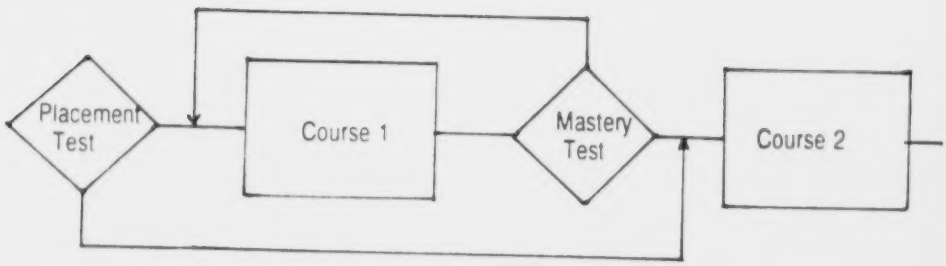
Case	Roots of $\lambda(x)$	
1	0.592	
2	0.094	
3	0.092	
4	0.654	5.455
5	0.442	2.778
6	0.746	

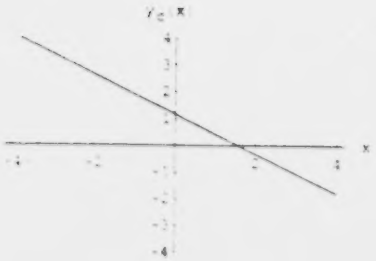
Figure Captions

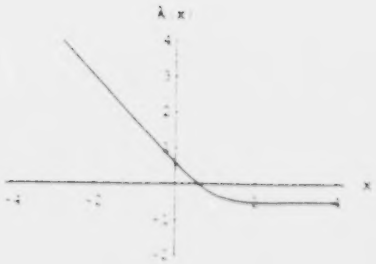
Figure 1. Flowchart of a two-course placement problem with a mastery test.

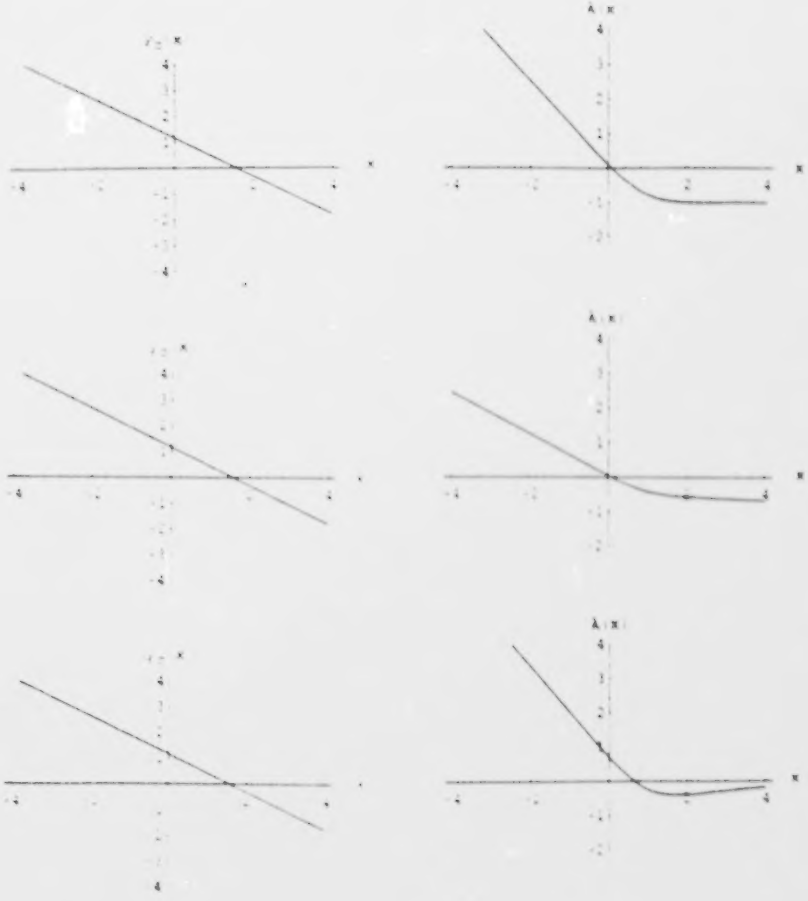
Figure 2. Optimal placement rule and mastery function for standard set of parameter values (Case 1).

Figure 3. Optimal placement rules and mastery functions for Cases 2 through 6.


Placement Test
Course 1
Mastery Test
Course 2







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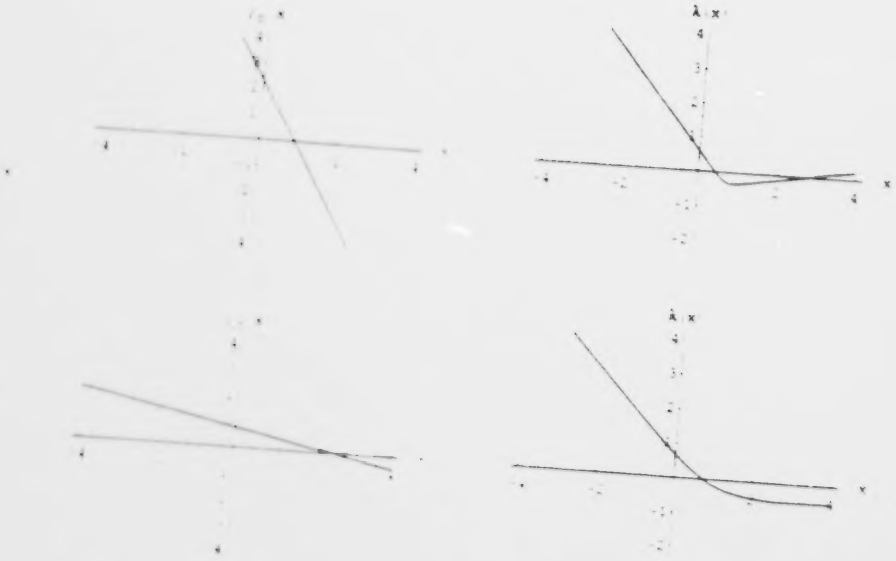

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